

The embedding of the spacetime in five-dimensional spaces with arbitrary non-degenerate Ricci tensor

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Abstract

We discuss and prove a theorem which asserts that any n -dimensional semi-Riemannian manifold can be locally embedded in a $(n+1)$ -dimensional space with a non-degenerate Ricci tensor which is equal, up to a local analytic diffeomorphism, to the Ricci tensor of an arbitrary specified space. This may be regarded as a further extension of the Campbell-Magaard theorem. We highlight the significance of embedding theorems of increasing degrees of generality in the context of higher dimensional spacetimes theories and illustrate the new theorem by establishing the embedding of a general class of Ricci-flat spacetimes.

I. INTRODUCTION

Modern physical theories which regard our spacetime as a hypersurface embedded in a five-dimensional manifold constitute nowadays a branch of theoretical physics undergoing quite an active research. On the other hand the idea of an extra fifth dimension is not new and goes back to the works of Kaluza and Klein carried out around the first quarter of the twentieth century [1,2]. Kaluza-Klein's seminal work has inspired theoretical physicists to generalize their conjecture in the construction of unified theories of the fundamental inter-

actions of nature. Subsequent developments which assume that the universe contains extra hidden dimensions include among others eleven-dimensional supergravity and superstring theories. [3,4]. More recently much attention has been devoted to the so-called Randall-Sundrum braneworld scenario where the spacetime is viewed as a four-dimensional hypersurface embedded in a five-dimensional Einstein space [5]. Non-compactified approaches to Kaluza-Klein gravity also makes use of embedding mechanisms and have been largely discussed in the literature [6].

In a sense one could say that all spacetime embedding theories [7] assume, implicitly or explicitly, a mathematical framework which must provide consistency for the postulates and basic principles set forth by such theories. In this connection it is of interest to know whether the embedding theorems of differential geometry are properly taken into account when constructing higher dimensional models. The analysis of the geometrical structure underlying some modern embedding theories has recently attracted the interest of some authors [8–13]. It seems that there is now a quest for embedding theorems with increasing degrees of generality, i.e., theorems ensuring that arbitrary n -dimensional spacetimes can be embedable in classes of $(n+1)$ -dimensional spaces the most general as possible.

Two theorems of historical importance which have played a significant role in physical theories of higher dimensions should be mentioned. The first is the well-known Janet-Cartan theorem, which asserts that if the embedding space is flat, the minimum number of extra dimensions needed to analytically embed a n -dimensional Riemannian manifold is d , with $0 \leq d \leq n(n-1)/2$. [14].

The second is a little known but powerful theorem due to Campbell [15] the proof of which was outlined by Campbell and completed by Magaard [16]. The content of the Campbell-Magaard theorem is that any n -dimensional Riemannian manifold with analytic metric, locally, can be isometrically embedded into a certain $(n+1)$ -dimensional Ricci-flat manifold. It is interesting to note that both theorems specify a geometry property to be satisfied by the embedding space by imposing the restrictions $R_{\mu\nu\lambda\rho} = 0$ in one case and $R_{\mu\nu} = 0$ in the other. It is also worth of noting that by relaxing the flatness condition, assumed in the

Janet-Cartan theorem, and replacing it by the weaker Ricci-flatness condition, the Campbell-Magaard theorem drastically reduces the codimension of the embedding space to $d = 1$. This seems to give support to the mathematical consistency of theories in which the dynamics of the embedding space is governed by the vacuum Einstein field equations [6]. However, the view adopted by Randall-Sundrum braneworld model [5] that the embedding space, i.e. the bulk, should correspond to an Einstein space sourced by a negative cosmological constant has naturally raised the question of whether Campbell-Magaard theorem could be extended to include embeddings in arbitrary Einstein spaces. This conjecture was shown to be, in fact, a theorem the proof of which is given in ref. [13]. Embeddings into spaces sourced by scalar fields also have been considered and a different extension of the Campbell-Magaard theorem has been proved [12,17]. In seeking higher levels of generalization one is led to consider the more general situation of embedding spaces whose Ricci tensor is arbitrary. In this paper we shall be concerned with this problem. In section II we state and prove a theorem which considers embedding spaces with arbitrary non-degenerate Ricci tensor, and, in a way, would represent a further generalization of Campbell-Magaard's result. In section III we illustrate the theorem by establishing the embedding of a general class of Ricci-flat spacetimes in a given collection of five-dimensional spaces whose Ricci tensor is equivalent to a specified non-degenerate and non-constant Ricci tensor.

We believe that insofar as five-dimensional embedding theories are metric theories it appears to be of relevance to allow the embedding spaces to have different geometrical properties, which must ultimately be determined by the dynamics of the theory in question. Therefore generalizations of the known embedding theorems might be helpful in building new higher dimensional models.

II. EXTENSION OF CAMPBELL-MAGAARD THEOREM: EMBEDDING SPACES WITH ARBITRARY NON-DEGENERATE RICCI TENSOR

In this section we want to investigate the existence of a local analytic embedding of a n -dimensional semi-Riemannian manifold (M^n, g) into a class of $(n + 1)$ -dimensional spaces whose Ricci tensor is *equivalent* to the Ricci tensor of a $(n + 1)$ -dimensional space arbitrarily specified.

Definition. Consider a $(n + 1)$ -dimensional semi-Riemannian space $(\tilde{M}_0^{n+1}, \tilde{g}_0)$ and let $S_{\alpha\beta}$ denote the components of the Ricci tensor in a coordinate system $\{x'^\alpha\}$. Let $(\tilde{M}^{n+1}, \tilde{g})$ be another $(n + 1)$ -dimensional semi-Riemannian space with $\tilde{R}_{\alpha\beta}$ denoting the components of the Ricci-tensor in a coordinate system $\{x^\alpha\}$ which covers a neighborhood of a point $p \in \tilde{M}^{n+1}$ whose coordinates are $x_p^1 = \dots = x_p^{n+1} = 0$. Then, we shall say that $S_{\alpha\beta}$ and $\tilde{R}_{\alpha\beta}$ are equivalent if there exists an analytic local diffeomorphism $\bar{f} : \tilde{M}_0^{n+1} \rightarrow \tilde{M}^{n+1}$ at p such that

$$\tilde{R}_{\alpha\beta}(x^\gamma) = \frac{\partial \bar{f}^\mu}{\partial x^\alpha} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu}(x'^\kappa), \quad (1)$$

where $x'^\kappa = \bar{f}^\kappa(x^\lambda)$. In others words, $S_{\alpha\beta}$ and $\tilde{R}_{\alpha\beta}$ are said to be equivalent if there exists a analytic function $\bar{f}^\mu = \bar{f}^\mu(x^\alpha)$ such that: i) $\left| \frac{\partial \bar{f}^\mu}{\partial x^\alpha} \right| \neq 0$ at $0 \in \mathbb{R}^{n+1}$; ii) the condition (1) holds in a neighborhood of $0 \in \mathbb{R}^{n+1}$. In this case, $(\tilde{M}_0^{n+1}, \tilde{g}_0)$ and $(\tilde{M}^{n+1}, \tilde{g})$ are said to be “Ricci-equivalent” spaces. ¹

Clearly, from the above, the collection $\mathcal{M}_{\tilde{g}_0}^{n+1}$ of all spaces which are Ricci-equivalent to a given space $(\tilde{M}_0^{n+1}, \tilde{g}_0)$ is well defined. Therefore it makes sense to discuss the existence of the embedding of a given arbitrary n -dimensional semi-Riemannian manifold (\tilde{M}^n, \tilde{g}) into the class $\mathcal{M}_{\tilde{g}_0}^{n+1}$. In what follows we shall show that if the Ricci tensor of $(\tilde{M}_0^{n+1}, \tilde{g}_0)$ is non-degenerate, i.e., the matrix formed by its components has inverse, then the existence

¹Henceforth we shall follow the convention adopted in ref [13] where Latin and Greek indices run from 0 to n and $n + 1$, respectively.

of the embedding can be ensured.

We should note, however, that (1) defines a notion of equivalence between the covariant Ricci tensor $S_{\alpha\beta}$ and $\tilde{R}_{\alpha\beta}$. This equivalence does not imply that the contravariant Ricci tensor $S^{\alpha\beta}$ and $\tilde{R}^{\alpha\beta}$ are also equivalent. In general they are not, unless the diffeomorphism is an isometry, a condition which is more restrictive than (1).

Let us consider $(\tilde{M}^{n+1}, \tilde{g})$ and choose a coordinate system in which the metric has the form

$$ds^2 = \bar{g}_{ik} dx^i dx^k + \varepsilon \bar{\phi}^2 dy^2, \quad (2)$$

where $\varepsilon = \pm 1$. In these coordinates (1) may be written in the following equivalent form:

$$\tilde{R}_{ik} = \bar{R}_{ik} + \varepsilon \bar{g}^{jm} (\bar{\Omega}_{ik} \bar{\Omega}_{jm} - 2 \bar{\Omega}_{jk} \bar{\Omega}_{im}) - \frac{\varepsilon}{\bar{\phi}} \frac{\partial \bar{\Omega}_{ik}}{\partial y} + \frac{1}{\bar{\phi}} \bar{\nabla}_i \bar{\nabla}_k \bar{\phi} = \frac{\partial \bar{f}^\mu}{\partial x^i} \frac{\partial \bar{f}^\nu}{\partial x^k} S_{\mu\nu} (\bar{f}^\alpha) \quad (3)$$

$$\tilde{R}_i^y = \frac{\varepsilon}{\bar{\phi}} \bar{g}^{jk} (\bar{\nabla}_j \bar{\Omega}_{ik} - \bar{\nabla}_i \bar{\Omega}_{jk}) = \frac{\varepsilon}{\bar{\phi}^2} \frac{\partial \bar{f}^\mu}{\partial y} \frac{\partial \bar{f}^\nu}{\partial x^i} S_{\mu\nu} (\bar{f}^\alpha) \quad (4)$$

$$\begin{aligned} \tilde{G}_y^y = -\frac{1}{2} \bar{g}^{ik} \bar{g}^{jm} (\bar{R}_{ijk m} + \varepsilon (\bar{\Omega}_{ik} \bar{\Omega}_{jm} - \bar{\Omega}_{jk} \bar{\Omega}_{im})) &= \frac{1}{2} \frac{\varepsilon}{\bar{\phi}^2} \frac{\partial \bar{f}^\mu}{\partial y} \frac{\partial \bar{f}^\nu}{\partial y} S_{\mu\nu} (\bar{f}^\alpha) \\ &\quad - \frac{1}{2} \bar{g}^{jm} \frac{\partial \bar{f}^\mu}{\partial x^j} \frac{\partial \bar{f}^\nu}{\partial x^m} S_{\mu\nu} (\bar{f}^\alpha), \end{aligned} \quad (5)$$

where

$$\bar{\Omega}_{ik} = -\frac{1}{2\bar{\phi}} \frac{\partial \bar{g}_{ik}}{\partial y}, \quad (6)$$

$G_{\alpha\beta}$ is the Einstein tensor and a bar is used to denote all the geometrical quantities calculated with the induced metric \bar{g}_{ik} on a generic hypersurface Σ_c of the foliation $y = c = \text{const.}$ Before we state the main theorem we need a few preliminaries.

We begin by defining the tensor

$$\tilde{F}_\beta^\alpha = \tilde{G}_\beta^\alpha - \left(\tilde{g}^{\alpha\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} - \frac{1}{2} \delta_\beta^\alpha \tilde{g}^{\gamma\lambda} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\lambda} S_{\mu\nu} \right). \quad (7)$$

If we now impose that the functions \bar{f}^α satisfy the equation

$$\tilde{\nabla}_\alpha \left(\tilde{g}^{\alpha\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} - \frac{1}{2} \delta_\beta^\alpha \tilde{g}^{\gamma\lambda} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\lambda} S_{\mu\nu} \right) = 0, \quad (8)$$

then it is easily seen that, as the Einstein tensor G_β^α has vanishing divergence, the tensor \tilde{F}_β^α also is divergenceless for any metric $\tilde{g}_{\alpha\beta}$, even those which are not solutions of Eq. (1). Thus, we are ready to state the following lemma.

Lemma 1. *Let the functions $\bar{g}_{ik}(x^1, \dots, x^n, y)$, $\bar{\phi}(x^1, \dots, x^n, y)$ and $\bar{f}^\alpha(x^1, \dots, x^n, y)$ be analytical at $(0, \dots, 0) \in \Sigma_0 \subset \mathbb{R}^{n+1}$. Assume that the following conditions hold*

- i) $\bar{g}_{ik} = \bar{g}_{ki}$;
- ii) $\det(\bar{g}_{ik}) \neq 0$;
- iii) $\bar{\phi} \neq 0$.

Suppose further that \bar{g}_{ik} and \bar{f}^α satisfy the equations (3) and (8) in an open set $V \subset \mathbb{R}^{n+1}$ which contains $0 \in \mathbb{R}^{n+1}$, and (4) and (5) hold at Σ_0 . Then, \bar{g}_{ik} , $\bar{\phi}$ and \bar{f}^α satisfy (4) and (5) in a neighborhood $0 \in \mathbb{R}^{n+1}$.

Proof. The key point of the proof is given by the equation $\tilde{\nabla}_\alpha \tilde{F}_\beta^\alpha = 0$, which can be written as

$$\frac{\partial \tilde{F}_\beta^y}{\partial y} = -\frac{\partial \tilde{F}_\beta^i}{\partial x^i} - \tilde{\Gamma}_{\mu\lambda}^\mu \tilde{F}_\beta^\lambda + \tilde{\Gamma}_{\lambda\beta}^\mu \tilde{F}_\mu^\lambda. \quad (9)$$

On the other hand, by assumption (3) holds in $V \subset \mathbb{R}^{n+1}$. Then, it can be shown that in V we have $\tilde{F}_k^i = -\delta_k^i \tilde{F}_y^y$. After some algebra we can deduce that

$$\frac{\partial \tilde{F}_y^y}{\partial y} = -\varepsilon \bar{\phi}^2 \bar{g}^{ij} \frac{\partial \tilde{F}_i^y}{\partial x^j} - 2\tilde{\Gamma}_{iy}^i \tilde{F}_y^y + \left(-\varepsilon \frac{\partial (\bar{\phi}^2 \bar{g}^{ij})}{\partial y^j} - \varepsilon \bar{\phi}^2 \bar{g}^{ij} \tilde{\Gamma}_{kj}^k + \tilde{\Gamma}_{yy}^i \right) \tilde{F}_i^y \quad (10)$$

$$\frac{\partial \tilde{F}_i^y}{\partial y} = \frac{\partial \tilde{F}_y^y}{\partial x^i} + 2\tilde{\Gamma}_{yi}^y \tilde{F}_y^y + \left(\tilde{\Gamma}_{yi}^k + \varepsilon \bar{\phi}^2 \bar{g}^{kj} \tilde{\Gamma}_{ij}^y - \tilde{\Gamma}_{y\mu}^\mu \delta_i^k \right) \tilde{F}_k^y. \quad (11)$$

Since at the hypersurface Σ_0 the equations (4) and (5) also hold, it follows that $\tilde{F}_\beta^y = 0$ at Σ_0 and hence $\left. \frac{\partial \tilde{F}_\beta^y}{\partial y} \right|_{y=0} = 0$. It is not difficult to show by mathematical induction that all the derivatives (to any order) of \tilde{F}_β^y vanish at $y = 0$. As \tilde{F}_β^y is analytic we conclude that $\tilde{F}_\beta^y = 0$ in an open set of \mathbb{R}^{n+1} . Hence, Eqs. (4) and (5), which are equivalent to $\tilde{F}_\beta^y = 0$, also hold in an open set of \mathbb{R}^{n+1} which includes the origin. This proves the lemma.

The question which now arises is: do Eqs. (3) and (8) admit solution? To answer this question we first note that (3) can be expressed in the following form

$$\begin{aligned} \frac{\partial^2 \bar{g}_{ik}}{\partial y^2} = & -2\varepsilon \bar{\phi}^2 \left(\frac{\partial \bar{f}^\mu}{\partial x^i} \frac{\partial \bar{f}^\nu}{\partial x^k} S_{\mu\nu} \left(\bar{f}^\alpha \right) \right) + \frac{1}{\bar{\phi}} \frac{\partial \bar{\phi}}{\partial y} \frac{\partial \bar{g}_{ik}}{\partial y} - \frac{1}{2} \bar{g}^{jm} \left(\frac{\partial \bar{g}_{ik}}{\partial y} \frac{\partial \bar{g}_{jm}}{\partial y} - 2 \frac{\partial \bar{g}_{im}}{\partial y} \frac{\partial \bar{g}_{jk}}{\partial y} \right) \\ & - 2\varepsilon \bar{\phi} \left(\frac{\partial^2 \bar{\phi}}{\partial x^i \partial x^k} - \frac{\partial \bar{\phi}}{\partial x^j} \bar{\Gamma}_{ik}^j \right) - 2\varepsilon \bar{\phi}^2 \bar{R}_{ik}. \end{aligned} \quad (12)$$

Second, let us rewrite Eq. (8) in the form

$$\begin{aligned} & \frac{\partial}{\partial x^\alpha} \left(\tilde{g}^{\alpha\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} - \frac{1}{2} \delta_\beta^\alpha \tilde{g}^{\gamma\lambda} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\lambda} S_{\mu\nu} \right) + \\ & + \tilde{\Gamma}_{\alpha\sigma}^\alpha \left(\tilde{g}^{\sigma\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} \right) - \tilde{\Gamma}_{\alpha\beta}^\sigma \left(\tilde{g}^{\alpha\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\sigma} S_{\mu\nu} \right) = 0. \end{aligned} \quad (13)$$

We now isolate the terms which contain second-order derivatives of \bar{f}^α with respect to y in the equation above. Putting $\beta = n + 1$ we obtain

$$\begin{aligned} \frac{\varepsilon}{\bar{\phi}^2} \frac{\partial^2 \bar{f}^\mu}{\partial y^2} \frac{\partial \bar{f}^\nu}{\partial x^i} S_{\mu\nu} = & -\frac{1}{2} \frac{\partial \bar{f}^\mu}{\partial y} \frac{\partial \bar{f}^\nu}{\partial y} \frac{\partial}{\partial y} \left(\frac{\varepsilon}{\bar{\phi}^2} S_{\mu\nu} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(\bar{g}^{jk} \frac{\partial \bar{f}^\mu}{\partial x^j} \frac{\partial \bar{f}^\nu}{\partial x^k} S_{\mu\nu} \right) - \frac{\partial}{\partial x^j} \left(\bar{g}^{jk} \frac{\partial \bar{f}^\mu}{\partial x^k} \frac{\partial \bar{f}^\nu}{\partial y} S_{\mu\nu} \right) \\ & - \tilde{\Gamma}_{\alpha\sigma}^\alpha \left(\tilde{g}^{\sigma\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} \right) + \tilde{\Gamma}_{\alpha\beta}^\sigma \left(\tilde{g}^{\alpha\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\sigma} S_{\mu\nu} \right). \end{aligned} \quad (14)$$

For $\beta = i$ we have

$$\begin{aligned} \frac{\varepsilon}{\bar{\phi}^2} \frac{\partial^2 \bar{f}^\mu}{\partial y^2} \frac{\partial \bar{f}^\nu}{\partial x^i} S_{\mu\nu} = & -\frac{\partial \bar{f}^\mu}{\partial y} \frac{\partial}{\partial y} \left(\frac{\varepsilon}{\bar{\phi}^2} \frac{\partial \bar{f}^\nu}{\partial x^i} S_{\mu\nu} \right) - \frac{\partial}{\partial x^j} \left(\bar{g}^{jk} \frac{\partial \bar{f}^\mu}{\partial x^j} \frac{\partial \bar{f}^\nu}{\partial x^i} S_{\mu\nu} \right) + \frac{1}{2} \frac{\partial}{\partial x^i} \left(\tilde{g}^{\sigma\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} \right) \\ & - \tilde{\Gamma}_{\alpha\sigma}^\alpha \left(\tilde{g}^{\sigma\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} \right) + \tilde{\Gamma}_{\alpha\beta}^\sigma \left(\tilde{g}^{\alpha\gamma} \frac{\partial \bar{f}^\mu}{\partial x^\gamma} \frac{\partial \bar{f}^\nu}{\partial x^\sigma} S_{\mu\nu} \right). \end{aligned} \quad (15)$$

Clearly, the right-hand side of (14) and (15) does not contain second-order derivatives of the functions \bar{f}^α and \bar{g}_{ik} with respect to y . Therefore, they are of the form

$$\frac{\partial^2 \bar{f}^\mu}{\partial y^2} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} = Q_\beta \left(\bar{f}^\lambda, \frac{\partial \bar{f}^\lambda}{\partial x^\sigma}, \frac{\partial^2 \bar{f}^\lambda}{\partial x^\sigma \partial x^i}, \bar{g}_{ik}, \frac{\partial \bar{g}_{ik}}{\partial x^\sigma} \right). \quad (16)$$

(Of course Q_β also depends on $\bar{\phi}$ and its derivatives, however this fact is not relevant for our present reasoning). Thus, assuming that $\left| \frac{\partial \bar{f}^\mu}{\partial x^\alpha} \right| \neq 0$ (we shall see later on that this assumption can always be made) we can write

$$\frac{\partial^2 \bar{f}^\mu}{\partial y^2} S_{\mu\nu} = \frac{\partial x^\beta}{\partial \bar{f}^\nu} Q_\beta \left(\bar{f}^\lambda, \frac{\partial \bar{f}^\lambda}{\partial x^\sigma}, \frac{\partial^2 \bar{f}^\lambda}{\partial x^\sigma \partial x^i}, \bar{g}_{ik}, \frac{\partial \bar{g}_{ik}}{\partial x^\sigma} \right). \quad (17)$$

If we suppose that $S_{\mu\nu}$ is invertible, i.e., there exists $(S^{-1})^{\nu\lambda}$ such that

$$S_{\mu\nu} (S^{-1})^{\nu\lambda} = \delta_\mu^\lambda, \quad (18)$$

then (8) can be put into the canonical form

$$\frac{\partial^2 \bar{f}^\mu}{\partial y^2} = P^\mu \left(\bar{f}^\lambda, \frac{\partial \bar{f}^\lambda}{\partial x^\sigma}, \frac{\partial^2 \bar{f}^\lambda}{\partial x^\sigma \partial x^i}, \bar{g}_{ik}, \frac{\partial \bar{g}_{ik}}{\partial x^\sigma} \right), \quad (19)$$

where each P^μ is analytic with respect to its arguments provided that $\left| \frac{\partial \bar{f}^\mu}{\partial x^\alpha} \right| \neq 0$, $|\bar{g}_{ik}| \neq 0$ and $\bar{\phi} \neq 0$.

It easy to see that the Cauchy-Kowalewski theorem (see Appendix) can be applied to the equations (19) and (12), which are equivalent to (3) and (8), respectively. According to the above-mentioned theorem, if an analytic function $\bar{\phi} \neq 0$ is chosen, then there exists a unique set of analytic functions \bar{g}_{ik} and \bar{f}^α that are solutions of (3) and (8) satisfying the initial conditions

$$\bar{g}_{ik} (x^1, \dots, x^n, 0) = g_{ik} (x^1, \dots, x^n) \quad (20)$$

$$\frac{\partial \bar{g}_{ik}}{\partial y} (x^1, \dots, x^n, 0) = -2\bar{\phi} (x^1, \dots, x^n, 0) \Omega_{ik} (x^1, \dots, x^n) \quad (21)$$

$$\bar{f}^\alpha (x^1, \dots, x^n, 0) = \xi^\alpha (x^1, \dots, x^n) \quad (22)$$

$$\frac{\partial \bar{f}^\alpha}{\partial y} (x^1, \dots, x^n, 0) = \eta^\alpha (x^1, \dots, x^n), \quad (23)$$

where g_{ik} , Ω_{ik} , ξ^α and η^α are analytic functions at the origin $0 \in \mathbb{R}^n$, and the following conditions hold: i) $\left| \frac{\partial \bar{f}^\mu}{\partial x^\alpha} \right|_0 \neq 0$; e ii) $|g_{ik}| \neq 0$. (Incidentally, we can easily verify that the condition (i) is satisfied by simply choosing: $\xi^i = x^i$; $\xi^{n+1} = 0$; $\eta^i = 0$ and $\eta^{n+1} = 1$. With this choice, $\left| \frac{\partial \bar{f}^\mu}{\partial x^\alpha} \right|_0 = 1$.)

We now are ready to state the following theorem.

Theorem 1 *Let M^n be a n -dimensional semi-Riemannian manifold with metric given by*

$$ds^2 = g_{ik} dx^i dx^k,$$

in a coordinate system $\{x^i\}$ of M^n . Let $p \in M^n$, have coordinates $x_p^1 = \dots = x_p^n = 0$. Then, M^n has a local isometric and analytic embedding (at the point p) in a $(n+1)$ -dimensional space $(\tilde{M}^{n+1}, \tilde{g})$ whose Ricci tensor is equivalent to the symmetric, analytic and non-degenerate tensor $S_{\mu\nu}$ if and only if there exist functions $\Omega_{ik}(x^1, \dots, x^n)$ ($i, k = 1, \dots, n$), $\xi^\alpha(x^1, \dots, x^n)$, $\eta^\alpha(x^1, \dots, x^n)$ ($\alpha = 1, \dots, n+1$) and $\phi(x^1, \dots, x^n) \neq 0$ that are analytic at $0 \in \mathbb{R}^n$, such that

$$\Omega_{ik} = \Omega_{ki} \quad (24)$$

$$g^{jk}(\nabla_j \Omega_{ik} - \nabla_i \Omega_{jk}) = \frac{1}{\phi} \eta^\mu \frac{\partial \xi^\nu}{\partial x^i} S_{\mu\nu}(\xi^\alpha) \quad (25)$$

$$g^{ik} g^{jm} (R_{ijkm} + \varepsilon (\Omega_{ik} \Omega_{jm} - \Omega_{jk} \Omega_{im})) = -\frac{\varepsilon}{\phi^2} \eta^\mu \eta^\nu S_{\mu\nu}(\xi^\alpha) + g^{jm} \frac{\partial \xi^\mu}{\partial x^j} \frac{\partial \xi^\nu}{\partial x^m} S_{\mu\nu}(\xi^\alpha) \quad (26)$$

$$\begin{vmatrix} \frac{\partial \xi^1}{\partial x^1} & \dots & \frac{\partial \xi^{n+1}}{\partial x^1} \\ \vdots & & \vdots \\ \frac{\partial \xi^1}{\partial x^n} & \dots & \frac{\partial \xi^{n+1}}{\partial x^n} \\ \eta^1 & \dots & \eta^{n+1} \end{vmatrix} \neq 0 \quad (27)$$

Proof. Let us start with the necessary condition. If (M^n, g) has an embedding in $(\tilde{M}^{n+1}, \tilde{g})$, then it can be proved that there exists a coordinate system in which the metric of the embedding space has the form [13]

$$ds^2 = \bar{g}_{ik} dx^i dx^k + \varepsilon \bar{\phi}^2 dy^2, \quad (28)$$

where the analytic functions $\bar{g}_{ik}(x^1, \dots, x^n, y)$ and $\bar{\phi}(x^1, \dots, x^n, y)$ are such that $\bar{\phi}(x^1, \dots, x^n, y) \neq 0$ and that $\bar{g}_{ik}(x^1, \dots, x^n, 0) = g_{ik}(x^1, \dots, x^n)$ in an open set of \mathbb{R}^n which contains the origin. Given that the Ricci tensor of the embedding space $(\tilde{M}^{n+1}, \tilde{g})$ is, by assumption, equivalent to $S_{\mu\nu}$, then the equations (3), (4), (5) and (8) are satisfied in a neighborhood of $0 \in \mathbb{R}^{n+1}$ for some functions \bar{f}^μ . In particular, the equations (4) and (5) hold for $y = 0$. Therefore, if we define Ω_{ik} , ξ^α , η^α by the relations (21), (22) and (23), and take $\phi(x^1, \dots, x^n) = \bar{\phi}(x^1, \dots, x^n, 0)$ then the Eqs.(24), (25), (26) and (27) are satisfied.

Let us turn to the sufficiency. Suppose there exist functions $\Omega_{ik}(x^1, \dots, x^n)$, $\xi^\alpha(x^1, \dots, x^n)$,

$\eta^\alpha(x^1, \dots, x^n)$ and $\phi(x^1, \dots, x^n) \neq 0$ which satisfy (24), (25), (26) and (27). Choose an analytic function $\bar{\phi}(x^1, \dots, x^n, y) \neq 0$ such that $\bar{\phi}(x^1, \dots, x^n, 0) = \phi(x^1, \dots, x^n)$. By virtue of the Cauchy-Kowalewski theorem there exists a unique set of analytic functions $\bar{g}_{ik}(x^1, \dots, x^n, y)$ and $\bar{f}^\alpha(x^1, \dots, x^n, y)$ satisfying the equations (3), (8) and the initial conditions (20), (21), (22) and (23). Since, by assumption, the initial conditions satisfy the equations (24), (25) and (26) then $\bar{g}_{ik}, \bar{\phi}$ and \bar{f}^α satisfy (4) and (5) at $y = 0$. It follows from lemma 1 that $\bar{g}_{ik}, \bar{\phi}$ and \bar{f}^α satisfy (1) in an open set of \mathbb{R}^{n+1} which contains the origin. Further we can say that \bar{f}^α is a diffeomorphism since by virtue (27) we have $\left| \frac{\partial \bar{f}^\mu}{\partial x^\alpha} \right| \neq 0$. Therefore, we conclude that the $(n+1)$ -dimensional manifold whose line element (28) is formed with the solutions \bar{g}_{ik} and $\bar{\phi}$ is a space whose Ricci tensor is equivalent to $S_{\mu\nu}$, and the embedding of the manifold (M^n, g) is given by $y = 0$. This completes the proof.

We now need to show that once the functions g_{ik} are given the system of equations (24), (25), (26) and (27) always admits solution for Ω_{ik} . For simplicity we take $\xi^i = x^i$; $\xi^{n+1} = 0$; $\eta^i = 0$ e $\eta^{n+1} = 1$. With this choice the condition (27) is readily satisfied. The equations (24), (25) and (26) constitute a set of n partial differential equations (25) plus a constraint equation (26) for $n(n+1)/2$ independent functions Ω_{ik} . Except for $n = 1$, the number of unknown functions is greater than (or equal to $(n=2)$) the number of equations. Then, out of the set of functions Ω_{ik} we pick n functions Ω_{1k} ($k \geq 2$) and $\Omega_{r'n}$ to be regarded as the unknowns². The next step is to write (25) in a suitable form for application of the Cauchy-Kowalewski theorem (first-order derivative version) to ensure the existence of the solution. For the sake of brevity we shall omit the detailed proof and refer the reader to references [13,16] where a similar procedure is carried out. Then it can be shown that after solving

²The r' index has the following meaning. We assume, for the sake of the argument, that we are using a coordinate system in which $g_{11} \neq 0$ and $g_{1k} = 0$, $k = 2, \dots, n$. Hence, there exists at least an index $r' > 1$ such that $g^{r'n} \neq 0$, since $|g_{ik}| \neq 0$.

(25) for Ω_{1k} ($k \geq 2$) and $\Omega_{r'n}$ we obtain

$$\begin{aligned} \frac{\partial \Omega_{r'n}}{\partial x^1} = \frac{1}{g^{r'n}(\delta_{r'n} - 2)} & \left[- \underset{r,s>1}{g^{rs}} \Omega_{1s,r} + 2 \underset{\substack{1<r<s \\ r,s \neq r',n}}{g^{rs}} \Omega_{rs,1} + \underset{\substack{r>1 \\ r \neq r'}}{g^{rr}} \Omega_{rr,1} + \right. \\ & \left. + g^{r'r'} \Omega_{r'r',1} (1 - \delta_{r'n}) - \underset{r,s>1}{g^{rs}} \left(\Omega_{tr} \Gamma_{s1}^t + \underset{r<t}{\Omega_{rt}} \Gamma_{s1}^t - \Omega_{11} \Gamma_{sr}^1 - \underset{t<1}{\Omega_{1t}} \Gamma_{sr}^t \right) + \frac{1}{\phi} S_{1y}(x^i) \right] \end{aligned} \quad (29)$$

where no sum over r' is implied, and

$$\begin{aligned} \frac{\partial \Omega_{1k}}{\partial x^1} = g_{11} & \left[- \underset{r,s>1}{g^{rs}} \left(\underset{s \leq k}{\Omega_{sk,r}} + \underset{k < s}{\Omega_{ks,r}} - 2 \underset{r < s}{\Omega_{rs,k}} \right) - g^{11} \Omega_{11,k} - \underset{r>1}{g^{rr}} \Omega_{rr,k} \right. \\ & \left. - g^{rs} \left(\underset{t \leq r}{\Omega_{tr}} \Gamma_{sk}^t + \underset{r < t}{\Omega_{rt}} \Gamma_{sk}^t - \underset{t \leq k}{\Omega_{tk}} \Gamma_{sr}^t - \underset{k < t}{\Omega_{kt}} \Gamma_{sr}^t \right) + \frac{1}{\phi} S_{ky}(x^i) \right], \quad k \geq 2. \end{aligned} \quad (30)$$

where Ω_{11} must be substituted by

$$\begin{aligned} \Omega_{11} = \frac{1}{2g^{11} \underset{r,s>1}{g^{rs}} \left(\underset{r \leq s}{\Omega_{rs}} + \underset{s < r}{\Omega_{sr}} \right)} & \left[2g^{11} \underset{r,s>1}{g^{rs}} \Omega_{1r} \Omega_{1s} \right. \\ & - \underset{r,s,t,u>1}{g^{rs} g^{tu}} \left[\left(\underset{r \leq s}{\Omega_{rs}} + \underset{s < r}{\Omega_{sr}} \right) \left(\underset{t \leq u}{\Omega_{tu}} + \underset{u < t}{\Omega_{ut}} \right) - \left(\underset{r \leq u}{\Omega_{ru}} + \underset{u < r}{\Omega_{ur}} \right) \left(\underset{s \leq t}{\Omega_{st}} + \underset{t < s}{\Omega_{ts}} \right) \right] \\ & \left. - \varepsilon \left(R - \frac{\varepsilon}{\phi^2} S_{yy}(x^i) + g^{jm} S_{jm}(x^i) \right) \right]. \end{aligned} \quad (31)$$

Finally, if we choose the functions Ω_{ik} ($i \leq k, i > 1, (i, k) \neq (r', n)$), $\phi \neq 0$ as being analytic at the origin, and since $S_{\mu\nu}(x^i)$ are also analytic, then in view of the Cauchy-Kowalewski theorem the system of equations (29) and (30) admits a solution that is analytic at the origin. Therefore, given arbitrary analytic functions $g_{ik}(x^1, \dots, x^n)$ the existence of the functions $\Omega_{ik}(x^1, \dots, x^n)$ ($i, k = 1, \dots, n$), $\xi^\alpha(x^1, \dots, x^n)$, $\eta^\alpha(x^1, \dots, x^n)$ which satisfy (24), (25), (26) and (27) is ensured, so Theorem 1 applies.

It should be mentioned that in the case where $S_{\mu\nu} = 0$, Eq. (8) holds for any functions $\bar{f}^\alpha(x^\beta)$; hence all the results derived above applies when the space $(\tilde{M}_0^{n+1}, \tilde{g}_0)$ has a vanishing Ricci tensor. Therefore, we can state the following theorem:

Theorem 2. *Let M^n be a n -dimensional semi-Riemannian manifold with metric given by*

$$ds^2 = g_{ik} dx^i dx^k,$$

in a coordinate system $\{x^i\}$ of M^n . Let $p \in M^n$, have coordinates $x_p^1 = \dots = x_p^n = 0$. Consider a $(n+1)$ -dimensional semi-Riemannian space $(\tilde{M}_0^{n+1}, \tilde{g}_0)$ whose Ricci tensor is either non-degenerate or null. If g_{ik} are analytic functions at $0 \in \mathbb{R}^n$, then (M^n, g) has a local isometric and analytic embedding (at the point p) in a $(n+1)$ -dimensional space which is Ricci-equivalent to $(\tilde{M}_0^{n+1}, \tilde{g}_0)$.

Therefore, we conclude that if the space $(\tilde{M}_0^{n+1}, \tilde{g}_0)$ is a solution of the Einstein equations for some source, then Theorem 2 guarantees that there exists a space which satisfies the “same” Einstein equations up to a coordinate transformation (see Eq. (1)), in which the spacetime (M^n, g) can be embedded

III. A SIMPLE APPLICATION OF THEOREM 2

Up to this point we have considered the Ricci tensor only through its covariant components $\tilde{R}_{\alpha\beta}$. However, it is not difficult to realize that all the previous results we have obtained are still valid if the mixed \tilde{R}_β^α or contravariant components $\tilde{R}^{\alpha\beta}$ are considered instead.

In what follows we illustrate Theorem 2 in its Ricci-tensor mixed-components version.

Consider the five-dimensional semi-Riemannian space $(\tilde{M}_0^5, \tilde{g}_0)$ with a metric given by

$${}^5ds^2 = (y+1)^{\frac{4}{5}} (-dt^2 + dx^2 + dy^2 + dz^2) + \frac{24}{25}\varepsilon dy^2. \quad (32)$$

If we calculate the mixed components of the Ricci S_ν^μ tensor for this metric we obtain

$$S_\nu^\mu = \text{diag} \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1 \right) \frac{\varepsilon}{(y+1)^2}. \quad (33)$$

We can view (32) as a five-dimensional analogue of the Friedmann-Robertson-Walker cosmological metric for radiation, with an energy density given by $\rho(y) = -\frac{\varepsilon}{(y+1)^2}$ (as measured by observers ∂_y).

Consider now a four-dimensional space (M^4, g) with a vanishing Ricci tensor, i.e., a vacuum solution of the Einstein field equations. Let us consider the question of embedding (M^4, g) into the collection $\mathcal{M}_{g_0}^5$ of five-dimensional spaces that are Ricci-equivalent to

$(\tilde{M}_0^5, \tilde{g}_0)$. In order to work with a mixed Ricci tensor we redefine the Ricci-equivalence property by the equation

$$\tilde{R}_\nu^\mu = \frac{\partial \bar{f}^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{f}^\nu} S_\beta^\alpha. \quad (34)$$

To find the embedding we begin with the ansatz

$$\bar{g}_{ik}(x^1, \dots, x^4, y) = u(y) g_{ik}(x^1, \dots, x^4) \quad (35)$$

$$\bar{\phi}(x^1, \dots, x^4, y) = 1 \quad (36)$$

$$\bar{f}^\alpha(x^1, \dots, x^4, y) = x^\alpha, \quad (37)$$

where g_{ik} is the metric of (M^4, g) and $u(y)$ is a function such that $u(0) = 1$.

From Lemma 1 we can show that (34) is equivalent to the ordinary differential equation

$$u' = \frac{4}{5} \frac{u}{(y+1)}. \quad (38)$$

Therefore, after integrating (38) we conclude that (M^4, g) has a local embedding in the space

$${}^5ds^2 = (y+1)^{\frac{4}{5}} (g_{ik} dx^i dx^k) + \frac{24}{25} \varepsilon dy^2, \quad (39)$$

whose Ricci tensor is the same as S_ν^μ , given by (33). Finally, it is worth mentioning that although the spaces (32) and (39) are Ricci-equivalent they are not isometric. This can simply be verified since the Weyl tensor $W_{\mu\nu\lambda\rho}$ calculated from (32) vanishes while (39) may have $W_{\mu\nu\lambda\rho} \neq 0$ for some g_{ik} (choose, for example, $g_{ik} dx^i dx^k$ to be the line element of Schwarzschild spacetime).

IV. FINAL COMMENTS

The restriction of the Ricci-tensor being non-degenerate, as required by Theorem 2, certainly imposes a limitation on the set of possible sources of the embedding space. For example, we would have to leave out of consideration solutions of the Einstein equations such as

cosmological models sourced by dust-type perfect fluid. We feel that although a great number of solutions of physical interest have non-degenerate Ricci-tensor, e.g. Friedman-Robertson-Walker models sourced by incoherent radiating perfect fluids, it seems indisputable that a theorem in which the condition of non-degeneracy is relaxed would be most welcome.

V. APPENDIX

Theorem (Cauchy-Kowalewski). *Let us consider the set of partial differential equations:*

$$\frac{\partial^2 u^A}{\partial (y^{n+1})^2} = F^A \left(y^\alpha, u^B, \frac{\partial u^B}{\partial y^\alpha}, \frac{\partial^2 u^B}{\partial y^\alpha \partial y^i}, \right), \quad A = 1, \dots, m \quad (40)$$

where u^1, \dots, u^m are m unknown functions of the $n+1$ variables y^1, \dots, y^n, y^{n+1} , $\alpha = 1, \dots, n+1$, $i = 1, \dots, n$, $B = 1, \dots, m$. Also, let $v^1, \dots, v^m, w^1, \dots, w^m$, functions of the variables y^1, \dots, y^n , be analytic at $0 \in \mathbb{R}^n$. If the functions F^A are analytic with respect to each of their arguments around the values evaluated at the point $y^1 = \dots = y^n = 0$, then there exists a unique solution of equations (40) which is analytic at $0 \in \mathbb{R}^{n+1}$ and that satisfies the initial condition

$$u^A(y^1, \dots, y^n, 0) = v^A(y^1, \dots, y^n) \quad (41)$$

$$\frac{\partial u^A}{\partial y^{n+1}}(y^1, \dots, y^n, 0) = w^A(y^1, \dots, y^n), \quad A = 1, \dots, m. \quad (42)$$

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